

Quantum Electron Transport in Disordered Wires with Symplectic Symmetry

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The conductance of disordered wires with symplectic symmetry is studied by the supersymmetric field theory. Special attention is focused on the case where the number of conducting channels is odd. Such a situation can be realized in metallic carbon nanotubes. The average dimensionless conductance $\langle g \rangle$ is obtained using Zirnbauer's super-Fourier analysis. It is shown that with increasing wire length, $\langle g \rangle \rightarrow 1$ in the odd-channel case, while $\langle g \rangle \rightarrow 0$ in the ordinary even-channel case. It should be emphasized that the so-called Zirnbauer's zero mode, which has been believed to be unphysical, is essential for describing the anomalous behavior in the odd-channel case.

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Phase-coherent electron transport in quasi-one-dimensional disordered wires is characterized by three universality classes.¹ The universality classes describe transport properties which are independent of the microscopic details of disordered wires. The orthogonal class consists of systems having time-reversal symmetry without spin-orbit scattering, while the unitary class is characterized by the absence of time-reversal symmetry. The systems having time-reversal symmetry with strong spin-orbit scattering belong to the symplectic class. The conductance as well as its moment averaged over an ensemble crucially depends on which universality class it belongs to. For example, weak-localization (weak-antilocalization) effect arises in the orthogonal (symplectic) class, while such effects disappear in the unitary class. We can systematically analyze such differences by using two different nonperturbative approaches.

The first is the scaling theory of Dorokhov,² and Mello, Pereyra and Kumar.³ Let us consider an ensemble of disordered wires of length L with N channels per spin. The conductance of a sample is expressed as $G = (2e^2/h)g$ with $g = \sum_{a=1}^N T_a$, where $\{T_a\}$ denote the transmission eigenvalues and the factor two in G corresponds to the electron spin. It is convenient to introduce $\lambda_a \equiv (1 - T_a)/T_a$. The statistical properties of g are determined by the probability distribution $P(\lambda_1, \lambda_2, \dots, \lambda_N, L)$. The scaling theory is based on an evolution equation for P with increasing L , which is called the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. The second is the supersymmetric field (SSF) theory of Efetov.⁴ In this approach, we express the conductance (or its moment) in terms of electron Green functions and rewrite its ensemble average in an integral form over matrices Q containing commuting and anticommuting variables. Consequently, the problem is reduced to a field theory with the one-dimensional nonlinear σ model. This approach enables us to calculate the average conductance and its variance by means of the super-Fourier analysis by Zirnbauer.^{5,6}

The equivalence of the two approaches had been questioned because, for the symplectic class, the Zirnbauer's

result indicates $\langle g \rangle \rightarrow 1/2$ as $L \rightarrow \infty$ while the DMPK equation gives $\langle g \rangle \rightarrow 0$.⁷ It had been recognized that the nonzero limiting value of $1/2$ is caused by the so-called Zirnbauer's zero mode, which appears in the super-Fourier analysis. This apparent controversy is resolved by Brouwer and Frahm.⁸ They pointed out that the zero mode must be excluded since its parity is incompatible with an appropriate initial condition. They showed that the super-Fourier analysis gives us the result which agrees with the DMPK approach if we exclude unphysical modes, including the zero mode, and correctly take Kramers degeneracy into account.

Recently, Ando and Suzuura⁹ showed that metallic carbon nanotubes (CNs) have an anomalous transport property, which cannot be explained within the ordinary universality classes. They studied CNs of length L when the potential range of scatterers is larger than the lattice constant, in the case where the Fermi level lies in several conducting channels. Note that the number of conducting channels is odd in CNs. They proved that one perfectly conducting channel is present even when $L \rightarrow \infty$. The present author¹⁰ derived the DMPK equation for this system based on the peculiar symmetry of the scattering matrix,⁹ and found that the system belongs to the symplectic class with an odd number of channels. A unit cell of a CN contains two sublattices, and this internal degree of freedom plays a role of a pseudospin. When the potential range of scatterers is larger than the lattice constant, there appears an additional symmetry, which drives the system into the symplectic class.¹¹ Thus, the symplectic symmetry in CNs has no connection with true spin as well as spin-orbit scattering. Previously, the transport property in the symplectic class has been considered only in the even-channel case.¹ However, the odd-channel case^{9,10,12,13} is very different from the ordinary even-channel case.

In this letter, we consider the average conductance of disordered wires with symplectic symmetry by means of the SSF theory. Our attention is focused on the case where the number N of conducting channels is odd. We calculate the average conductance in the long- L limit by

using the super-Fourier analysis. In this approach, we expand the so-called heat kernel in terms of the eigenfunctions of the Laplacian in the space in which the σ model is defined. We point out that appropriate eigenmodes for the expansion differ between the even- and odd-channel cases. Consequently, unphysical modes in the even-channel case turn out to be appropriate in the odd-channel case. It is shown that the Zirnbauer's zero mode, which must be excluded in the even-channel case, becomes essential in describing the anomalous behavior in the odd-channel case. If we include the appropriate modes and correctly take Kramers degeneracy into account, we obtain $\langle g \rangle \sim 1 + 2e^{-4L/\xi}$ in the odd-channel case when $L \gg \xi$, where $\xi \equiv 2(N-1)l$ with the mean free path l . The result is in agreement with that obtained from the DMPK equation.¹⁰ We hereafter treat only the symplectic class, and do not consider true spin. Our argument closely follows that by Brouwer and Frahm.⁸ We first introduce a generating function F for conductance, and show that the averaged F obtained by the SSF approach becomes equivalent to that by the DMPK approach. By inspecting this argument, we show which eigenmodes are appropriate in expanding the heat kernel and then obtain the average conductance.

If the number of the conducting channels is even (i.e., $N = 2m$), the dimensionless conductance g is expressed as $g = 2 \sum_{a=1}^m (1 + \lambda_a)^{-1}$, where the factor two represents Kramers degeneracy. Extending Rejzai's idea,¹⁴ Brouwer and Frahm⁸ introduced the generating function,

$$F_0(\hat{\theta}, \hat{\lambda}) = \prod_{a=1}^m f(\hat{\theta}, \lambda_a) \quad (1)$$

with

$$f(\hat{\theta}, \lambda) = \frac{(1 + 2\lambda + \cos(\theta_3 + \theta_4))(1 + 2\lambda + \cos(\theta_3 - \theta_4))}{(1 + 2\lambda + \cosh \theta_1)^2}. \quad (2)$$

Here, $\hat{\theta} = (\theta_1, \theta_3, \theta_4)$ and $\hat{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$. Once the ensemble average of $F_0(\hat{\theta}, \hat{\lambda})$ is presented, we can calculate the average conductance, its variance and the density of transmission eigenvalues.⁸ When the number of channels is odd (i.e., $N = 2m+1$), only one eigenvalue is not degenerate. We assign the N th eigenvalue for it without loss of generality and obtain $g = (1 + \lambda_N)^{-1} + 2 \sum_{a=1}^m (1 + \lambda_a)^{-1}$. In this case, we consider the following generating function

$$F(\hat{\theta}, \hat{\lambda}) = f(\hat{\theta}, \lambda_N)^{\frac{1}{2}} \times F_0(\hat{\theta}, \hat{\lambda}), \quad (3)$$

instead of eq. (1).

Before presenting the SSF treatment, we derive the evolution equation for the averaged generating function using the DMPK equation.¹⁰ The ensemble average of F is given by

$$\begin{aligned} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}} &= \int_0^\infty \prod_{a=1}^m d\lambda_a \int_0^\infty d\lambda_N \\ &\times \delta(\lambda_N) F(\hat{\theta}, \hat{\lambda}) P(\hat{\lambda}, L), \end{aligned} \quad (4)$$

where $P(\hat{\lambda}, L) = P(\lambda_1, \lambda_2, \dots, \lambda_m, L)$. The probability

distribution obeys the DMPK equation,¹⁰

$$\begin{aligned} l(N-1) \frac{\partial}{\partial L} P(\hat{\lambda}, L) \\ = \sum_{a=1}^m \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) J(\hat{\lambda}) \frac{\partial}{\partial \lambda_a} J^{-1}(\hat{\lambda}) P(\hat{\lambda}, L) \end{aligned} \quad (5)$$

with the mean free path l and

$$J(\hat{\lambda}) = \prod_{c=1}^m \lambda_c^2 \times \prod_{a=2}^m \prod_{b=1}^{a-1} |\lambda_a - \lambda_b|^4. \quad (6)$$

From eq. (5), we obtain the evolution equation,

$$l(N-1) \frac{\partial}{\partial L} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}} = \langle D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}}, \quad (7)$$

where

$$D_{\hat{\lambda}} = J^{-1}(\hat{\lambda}) \sum_{a=1}^m \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) J(\hat{\lambda}) \frac{\partial}{\partial \lambda_a}. \quad (8)$$

In the limit of $L \rightarrow 0$, all the channels become perfectly conducting (i.e., $\lambda_a = 0$ for any a). This ballistic initial condition results in

$$\lim_{L \rightarrow 0} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}} = \left(\frac{\cos \theta_3 + \cos \theta_4}{1 + \cosh \theta_1} \right)^{2m+1}. \quad (9)$$

In the even-channel case, the exponent $2m+1$ is replaced by $2m$.

We next derive the evolution equation by using the SSF theory based on the model by Iida, Weidenmüller and Zuk.^{15,16} The derivation has been presented by Brouwer and Frahm,⁸ so we only summarize it briefly. We assume that a quasi-one-dimensional wire consists of K small grains in series. Each grain has $2M$ electron states including Kramers degeneracy. The wire is coupled to two ideal leads. We assume that the left (right) lead contains N_1 (N_2) conducting channels. The Hamiltonian H of our system is described by the matrix elements $H_{\mu s_\mu \nu s_\nu}^{ij}$, where i, j denote the grains ($1 \leq i, j \leq K$), μ, ν denote the electron states ($1 \leq \mu, \nu \leq M$) and s_μ, s_ν denote the pseudospins. The elements $H_{\mu s_\mu \nu s_\nu}^{ii}$, which describe the electron states in the i th grain, are independent Gaussian random variables with zero mean values and

$$\begin{aligned} \overline{H_{\mu s_\mu \nu s_\nu}^{ii} H_{\mu' s_{\mu'} \nu' s_{\nu'}}^{ii}} &= \frac{v_1^2}{2M} \left(\delta_{\mu \nu'} \delta_{\nu \mu'} \delta_{s_\mu s_{\nu'}} \delta_{s_\nu s_{\mu'}} \right. \\ &\quad \left. - \delta_{\mu \mu'} \delta_{\nu \nu'} (i\sigma_y)_{s_{\mu'}, s_\mu} (i\sigma_y)_{s_\nu, s_{\nu'}} \right), \end{aligned} \quad (10)$$

where σ_y is the Pauli matrix. The coupling between the adjacent grains are described by $H_{\mu s_\mu \nu s_\nu}^{ij}$ ($j = i \pm 1$), which are another set of Gaussian random variables with zero mean values and

$$\begin{aligned} \overline{H_{\mu s_\mu \nu s_\nu}^{ij} H_{\mu' s_{\mu'} \nu' s_{\nu'}}^{ji}} &= \frac{v_2^2}{2M} \delta_{\mu \nu'} \delta_{\nu \mu'} \delta_{s_\mu s_{\nu'}} \delta_{s_\nu s_{\mu'}}, \\ \overline{H_{\mu s_\mu \nu s_\nu}^{ij} H_{\mu' s_{\mu'} \nu' s_{\nu'}}^{ij}} &= -\frac{v_2^2}{2M} \delta_{\mu \mu'} \delta_{\nu \nu'} (i\sigma_y)_{s_{\mu'}, s_\mu} (i\sigma_y)_{s_\nu, s_{\nu'}}. \end{aligned} \quad (11)$$

All other matrix elements $H_{\mu s_\mu \nu s_\nu}^{ij}$ vanish. The coupling to the leads is described by a fixed $2KM \times (N_1 + N_2)$ matrix W . Its elements $W_{\mu s_\mu a}^i$, where a denotes the modes in

the leads, are nonzero only for $i = 1$ and $1 \leq a \leq N_1$, and $i = K$ and $N_1 + 1 \leq a \leq N_1 + N_2$. We assume that the orthogonality relation $\pi \sum_{\mu s_\mu} W_{a\mu s_\mu}^i W_{\mu s_\mu b}^i = v_1 x \delta_{ab}$ holds for $i = 1, K$.¹⁵ We decompose W as $W = W_1 + W_2$, where W_1 and W_2 contain the block with $\{W_{\mu s_\mu a}^1\}$ and that with $\{W_{\mu s_\mu a}^K\}$, respectively. Brouwer and Frahm proved that the generating function eq. (3) can be expressed in the form,

$$F(\hat{\theta}, \hat{\lambda}) = \text{Sdet}^{-\frac{1}{2}} \left(E_F 1_8 - H 1_8 + i\pi W_1 W_1^\dagger Q + i\pi W_2 W_2^\dagger \Lambda \right), \quad (13)$$

where Sdet stands for the superdeterminant of a supermatrix and E_F the Fermi energy. The supermatrix Q is expressed as $Q = T^{-1} \Lambda T$ with $\Lambda = \text{diag}(1_4, -1_4)$ and

$$T = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \exp \begin{pmatrix} 0 & \frac{1}{2} \hat{\theta} \\ \frac{1}{2} \hat{\theta} & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad (14)$$

where u and v are 4×4 unitary supermatrices and

$$\hat{\theta} = \begin{pmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & i\theta_3 & i\theta_4 \\ 0 & 0 & i\theta_4 & i\theta_3 \end{pmatrix}. \quad (15)$$

We can rewrite F in terms of a Gaussian integral over supervector ψ as

$$F(\hat{\theta}, \hat{\lambda}) = \int D\psi \exp \left[\frac{i}{2} \psi^\dagger \Lambda \left(E_F 1_8 - H 1_8 + i\pi W_1 W_1^\dagger Q + i\pi W_2 W_2^\dagger \Lambda + i\epsilon \Lambda \right) \psi \right], \quad (16)$$

where ϵ is a positive infinitesimal. After performing the standard manipulations,^{4,6,15} we find that in the limit $M \rightarrow \infty$, the averaged generating function is given by

$$\langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}} = \int dQ_1 \int dQ_K f_1(Q, Q_1) f_2(\Lambda, Q_K) \times W(Q_1, Q_K), \quad (17)$$

where

$$W(Q_1, Q_K) = \int \prod_{j=2}^{K-1} dQ_j \times \exp \left(-(v_2/v_1)^2 \sum_{k=1}^{K-1} \text{Str}(Q_k Q_{k+1}) \right), \quad (18)$$

$$f_1(Q, Q_1) = \exp \left(-\frac{N_1}{2} \text{Str} \ln(1 + x Q Q_1) \right), \quad (19)$$

$$f_2(Q, Q_K) = \exp \left(-\frac{N_2}{2} \text{Str} \ln(1 + x Q Q_K) \right). \quad (20)$$

Here, Str denotes the supertrace for a supermatrix. Assuming that $v_1^2 \ll v_2^2$, we replace the sum in eq. (18) with an integral.^{6,15} Accordingly, the discrete number k is reduced to the continuous variable s . The propagator $W(Q, Q')$, called the heat kernel, obeys the heat equation in the supersymmetric space. Noting that $f_1(Q, Q')$

has the same symmetry as the heat kernel, we eventually find that $\langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}}$ obeys⁸

$$8(v_2/v_1)^2 \frac{\partial}{\partial s} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}} = \Delta_{\hat{\theta}} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}} \quad (21)$$

with the initial condition

$$\lim_{s \rightarrow 0} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}} = \int dQ' f_1(Q, Q') f_2(\Lambda, Q'). \quad (22)$$

The Laplacian is defined as

$$\Delta_{\hat{\theta}} = \sum_{j=1,3,4} J^{-1}(\hat{\theta}) \frac{\partial}{\partial \theta_j} J(\hat{\theta}) \frac{\partial}{\partial \theta_j} \quad (23)$$

with

$$J(\hat{\theta}) = \sin \theta_3 \sin \theta_4 \sinh^3 \theta_1 \times \prod_{s_1, s_2 = \pm 1} \sinh^{-2} \left[\frac{1}{2} (\theta_1 + i s_1 \theta_3 + i s_2 \theta_4) \right]. \quad (24)$$

We prove that the SSF approach is equivalent to the DMPK approach. It is convenient to introduce $D_{\hat{\lambda}}^{(0)}$, which appears in the evolution equation for the even-channel case,⁸ given by

$$D_{\hat{\lambda}}^{(0)} = J_0^{-1}(\hat{\lambda}) \sum_{a=1}^m \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) J_0(\hat{\lambda}) \frac{\partial}{\partial \lambda_a} \quad (25)$$

with $J_0(\hat{\lambda}) = \prod_{a=2}^m \prod_{b=1}^{a-1} |\lambda_a - \lambda_b|^4$. Using the relation $D_{\hat{\lambda}}^{(0)} F_0(\hat{\theta}, \hat{\lambda}) = \Delta_{\hat{\theta}} F_0(\hat{\theta}, \hat{\lambda})$, derived by Brouwer and Frahm, we obtain

$$\Delta_{\hat{\theta}} F(\hat{\theta}, \hat{\lambda}) = \left(D_{\hat{\lambda}}^{(0)} + \sum_{a=1}^m \delta g(\hat{\theta}, \lambda_a) \right) F(\hat{\theta}, \hat{\lambda}) + \left(\Delta_{\hat{\theta}} f(\hat{\theta}, \lambda_N)^{\frac{1}{2}} \right) F_0(\hat{\theta}, \hat{\lambda}), \quad (26)$$

where

$$\delta g(\hat{\theta}, \lambda) = \sum_{j=1,3,4} 2 \frac{\partial \ln f(\hat{\theta}, \lambda_N)^{\frac{1}{2}}}{\partial \theta_j} \frac{\partial \ln f(\hat{\theta}, \lambda)}{\partial \theta_j}. \quad (27)$$

We can show that $\Delta_{\hat{\theta}} f(\hat{\theta}, \lambda_N)^{1/2}|_{\lambda_N=0} = 0$ and

$$\left(D_{\hat{\lambda}}^{(0)} + \sum_{a=1}^m \delta g(\hat{\theta}, \lambda_a) \right) F(\hat{\theta}, \hat{\lambda})|_{\lambda_N=0} = D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda})|_{\lambda_N=0}, \quad (28)$$

which result in

$$\Delta_{\hat{\theta}} F(\hat{\theta}, \hat{\lambda})|_{\lambda_N=0} = D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda})|_{\lambda_N=0}. \quad (29)$$

Using this relation, we obtain

$$\begin{aligned} l(N-1) \frac{\partial}{\partial L} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}} &= \int_0^\infty \prod_{a=1}^m d\lambda_a \int_0^\infty d\lambda_N \delta(\lambda_N) P(\hat{\lambda}, L) D_{\hat{\lambda}} F(\hat{\theta}, \hat{\lambda}) \\ &= \int_0^\infty \prod_{a=1}^m d\lambda_a \int_0^\infty d\lambda_N \delta(\lambda_N) P(\hat{\lambda}, L) \Delta_{\hat{\theta}} F(\hat{\theta}, \hat{\lambda}) \\ &= \Delta_{\hat{\theta}} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}}. \end{aligned} \quad (30)$$

Thus, the evolution equation for $\langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{DMPK}}$ is equivalent to that for $\langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}}$ if we set

$$\frac{8}{s} \left(\frac{v_2}{v_1} \right)^2 = \frac{l(N-1)}{L} \equiv \frac{\xi}{2L}. \quad (31)$$

Now we focus on the initial condition. We assume that the coupling to the two leads is ideal, which is achieved by setting $x = 1$. Brouwer and Frahm showed that the ballistic initial condition for the DMPK approach (i.e., $\lambda_a = 0$ for any a) can be reproduced if we fix N_1 and take the limit $N_2 \rightarrow \infty$. We trace the same steps. The limit $N_2 \rightarrow \infty$ results in $f_2(\Lambda, Q') \rightarrow \delta(Q', \Lambda)$. Thus, the right-hand side of eq. (22) is reduced to $f_1(Q, \Lambda)$. We finally obtain

$$\begin{aligned} \lim_{s \rightarrow 0} \langle F(\hat{\theta}, \hat{\lambda}) \rangle_{\text{SSF}} &= \exp \left(-\frac{N_1}{2} \text{Str} \ln(1 + Q\Lambda) \right) \\ &= \left(\frac{\cos \theta_3 + \cos \theta_4}{1 + \cosh \theta_1} \right)^{N_1}. \end{aligned} \quad (32)$$

This is equivalent to eq. (9) if we set $N_1 = N = 2m + 1$. Thus, the SSF approach is equivalent to the DMPK approach if we accept the treatments described above. This indicates that the average conductance as well as its variance is the same whether computed from the SSF approach or from the DMPK approach. It should be emphasized that eq. (32) becomes the initial condition for the even-channel case if we set $N_1 = 2m$.

We here briefly summarize the Zirnbauer's super-Fourier analysis^{5,6} for the thick-wire limit which is defined by $N \rightarrow \infty$, $L/l \rightarrow \infty$ with a fixed Nl/L . In this analysis, we expand the heat kernel $W(Q, Q')$ in terms of the eigenfunctions of $\Delta_{\hat{\theta}}$. The eigenvalue equation is^{5,6,8}

$$\Delta_{\hat{\theta}} \phi_{\lambda, n_1, n_2}(\hat{\theta}) = -\frac{1}{4} \varepsilon(\lambda, n_1, n_2) \phi_{\lambda, n_1, n_2}(\hat{\theta}). \quad (33)$$

The eigenvalues are $\varepsilon(\lambda, n_1, n_2) = \lambda^2 + n_1^2 + n_2^2 - 1$, where $n_1, n_2 = 1, 3, 5, \dots$ and λ is a positive real number. In addition, there arises the zero mode $\varepsilon(i, 1, 1) = 0$, and the subsidiary series $\varepsilon(i, n, n-2) = 2(n-1)^2$, where $n = 3, 5, 7, \dots$. Using the resulting expansion, Zirnbauer obtained the expression of the average conductance,

$$\begin{aligned} \langle g \rangle_{\text{Zirnbauer}} &= T(i, 1, 1)/2 \\ &+ \sum_{n=3,5,7,\dots} (T(i, n, n-2) + T(i, n-2, n)) / 2 \\ &+ 2^4 \sum_{n_1, n_2=1,3,5,\dots} \int_0^\infty d\lambda \lambda (\lambda^2 + 1) \tanh(\pi\lambda/2) \\ &\times n_1 n_2 (\lambda^2 + n_1^2 + n_2^2 - 1) \\ &\times \prod_{\sigma, \sigma_1, \sigma_2=\pm 1} (-1 + i\sigma\lambda + \sigma_1 n_1 + \sigma_2 n_2)^{-1} \\ &\times T(\lambda, n_1, n_2) \end{aligned} \quad (34)$$

with $T(\lambda, n_1, n_2) = e^{-(\lambda^2 + n_1^2 + n_2^2 - 1)(L/2\xi)}$. This indicates that $\langle g \rangle \rightarrow 1/2$ when $L \rightarrow \infty$.^{5,6} This apparently contradicts the DMPK approach, which gives $\langle g \rangle \rightarrow 0$ in the even-channel case¹ and $\langle g \rangle \rightarrow 1$ in the odd-channel case.¹⁰

Note that the difference between the even- and odd-channel cases is not reflected in the Zirnbauer's argument. We show that the even-odd difference can be taken into account through the initial condition for the heat kernel. Let us consider an operation \hat{P} defined by

$$\hat{P}\phi(\theta_1, \theta_3, \theta_4) = \phi(\theta_1, \theta_3 + \pi, \theta_4 + \pi), \quad (35)$$

which is introduced by Brouwer and Frahm.⁸ The Laplacian $\Delta_{\hat{\theta}}$ commutes with \hat{P} , and the eigenfunctions have either even or odd parity. The even (odd) parity means that the eigenvalue of \hat{P} is $+1$ (-1). Zirnbauer considered the limit $N_1 = N_2 \rightarrow \infty$. If the same limit is taken in our argument, we find that $\langle F \rangle_{\text{SSF}} = W(Q, \Lambda)$ holds⁸ since $f_i(Q, Q')$ is reduced to the delta function $\delta(Q, Q')$ when $N_i \rightarrow \infty$. This indicates that we can obtain the heat kernel by expanding $\langle F \rangle_{\text{SSF}}$ in terms of the eigenfunctions of $\Delta_{\hat{\theta}}$ and then taking the limit $N_1 \rightarrow \infty$. By employing this method, we can obtain $W(Q, \Lambda)$ which reflects the even-odd difference. The crucial point is that before taking the limit $N_1 \rightarrow \infty$, the parity of eq. (32) is well defined; its parity is odd (even) when $N_1 = 2m + 1$ ($N_1 = 2m$). Accordingly, we must select only the eigenmodes having odd parity in the odd-channel case, while the even-parity modes are appropriate in the even-channel case. However, once the limit of $N_1 \rightarrow \infty$ is taken, eq. (32) is reduced to

$$\lim_{s \rightarrow 0} \langle F \rangle_{\text{SSF}} = \delta(Q, \Lambda), \quad (36)$$

which is equivalent to the Zirnbauer's initial condition, and we cannot derive such a parity-selection rule from it. Thus, the even-odd difference disappears. In order to correctly calculate the conductance, we must take care of this parity-selection rule. Brouwer and Frahm pointed out that the odd-parity modes are unphysical in the even-channel case. However, they were not aware that the odd-parity modes become essential in the odd-channel case.

In the even-channel case, the above argument tells us that we must neglect the eigenfunctions with odd parity. The zero mode as well as its subsidiary modes has odd parity,⁸ so they must be excluded. By omitting all the odd-parity modes and correctly taking account of Kramers degeneracy, Brouwer and Frahm obtained

$$\langle g \rangle \sim \frac{32}{9} \left(\frac{\pi\xi}{2L} \right)^{\frac{3}{2}} e^{-\frac{L}{2\xi}}, \quad (37)$$

which agrees with the DMPK approach.¹ In contrast, we must exclude the even-parity modes in the odd-channel case. This is the crucial difference between the ordinary even-channel case and our odd-channel case. If we omit all the even-parity modes and take Kramers degeneracy into account, we obtain

$$\begin{aligned} \langle g \rangle &= T(i, 1, 1) + \sum_{n=3,5,7,\dots} (T(i, n, n-2) + T(i, n-2, n)) \\ &+ 2^5 \sum_{\substack{n_1, n_2=1,3,5,\dots \\ n_1+n_2 \equiv 0 \pmod{4}}} \int_0^\infty d\lambda \lambda (\lambda^2 + 1) \tanh(\pi\lambda/2) \\ &\times n_1 n_2 (\lambda^2 + n_1^2 + n_2^2 - 1) \\ &\times \prod_{\sigma, \sigma_1, \sigma_2=\pm 1} (-1 + i\sigma\lambda + \sigma_1 n_1 + \sigma_2 n_2)^{-1} \end{aligned}$$

$$\times T(\lambda, n_1, n_2). \quad (38)$$

When $L \rightarrow \infty$, the leading contribution to $\langle g \rangle$ arises from the zero mode, which yields $\langle g \rangle \rightarrow 1$. This indicates the presence of one perfectly conducting channel. The deviation $\langle \delta g \rangle \equiv \langle g \rangle - 1$ from the quantized value is

$$\langle \delta g \rangle \sim 2e^{-\frac{4L}{\xi}}, \quad (39)$$

which comes from the subsidiary modes with $n = 3$. This clearly indicates that the zero mode as well as the subsidiary modes is essential in describing the anomalous transport property in the odd-channel case. The characteristic length scale for $\langle \delta g \rangle$ is $\xi_{\text{odd}} = \xi/4$, while that in the even-channel case is obtained as $\xi_{\text{even}} = 2\xi$ from eq. (37). Thus, ξ_{odd} is a factor 8 shorter than ξ_{even} . This result is in quantitative agreement with the DMPK approach.¹⁰

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